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An application of the algebraic generator coordinate method to Helmholtz Lie optics

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Abstract. In this paper a Hilbert space of the Helmholtz equation solutions has been constructed by means of the algebraic generator coordinate method. This Hilbert space has been regarded as a space of wavized Helmholtz Lie optics, which is a good model of systems with bounded momenta.

1. Introduction

The Helmholtz equation, known for more than a hundred years, is used in many fields of physics and is still an interesting object of investigation. Lie optics, in general, is an attractive and valuable approach to different areas of light and magnetic optics, signal analysis and tomography (for a review and references see [1,2]). On the other hand, Euclidean Lie optics [3] is a good physical and group theoretical model of systems with bounded momenta, very useful for analysis of the Helmholtz equation arising in this model in a very natural way.

The main purpose of this paper is to find a Hilbert space, denoted by $\mathcal{K}$, of functions suited for Helmholtz equation solutions applying the algebraic generator coordinate method (AGCM) to Euclidean Lie optics. This realization of the Hilbert space of Helmholtz equation solutions is unitarily equivalent to the realizations found in [4,5] but also allows other representations of the solutions because the scalar product in this Hilbert space is expressed by integrals over the whole space $\mathbb{R}^3$ and not only over the space of screen variables as in [3]. It is useful in many applications.

The class of spaces constructed in this paper also furnishes the carrier spaces of some irreducible and unitary representations of the Euclidean group $\mathcal{E}_3 = ISO(3)$.

Another interesting problem analysed in this paper is consideration of the space $\mathcal{K}$ as a possible space of wave states for Euclidean (Helmholtz) Lie optics.

In section 1 we collect the most important information about Euclidean Lie optics together with the underlying group structures and the specific model, the so-called Helmholtz Lie optics, which are needed for further consideration. In section 3, for convenience, we adapt the AGCM to the case of Euclidean Lie optics. Section 4 contains the main result of the work—the construction of the optical state space for Helmholtz optics which is irreducible with respect to $\mathcal{E}_3$ and which is a Hilbert space of the Helmholtz equation solutions. In this section we also discuss some aspects of a wavization procedure for Helmholtz optics.

2. Euclidean Lie optics

The phase space of Euclidean Lie optics, improbable in point-particle mechanics, is bounded in momenta which range over the so called Descartes sphere of radius n (n denotes the refractive index) and the basic group of global optics is the three-dimensional Euclidean group $\mathcal{E}_3 = ISO(3) = SO(3)[T^3]$ of rigid motions in $\mathbb{R}^3$ which is the semisimple product of the rotational group $SO(3)$ and the translational group in three dimensions T^3 [3]. Following the notation presented in [3] the elements of the Euclidean group, which is the group of motion of this optics, can be parametrized as follows

$$\mathcal{E}_3 \ni E(\Omega, v) = E(\Omega, 0)E(1, v) \tag{1}$$

where Ω is a proper orthogonal 3×3 matrix which depicts rotation in $\mathbb{R}^3$ and v is a Cartesian three-dimensional vector responsible for translations. The group multiplication is given by the following relations:

(i) $E(\Omega_1, v_1)E(\Omega_2, v_2) = E(\Omega_1\Omega_2, v_1\Omega_2 + v_2)$,

(ii) the group unit is $E(1, 0)$,

(iii) the inverse element is $E(\Omega, v)^{-1} = E(\Omega^{-1}, -v\Omega^{-1})$.

On the six-dimensional Euclidean group manifold one can build a space of functions $f(P, r)$, where the dependence on P is referred to the direction in $\mathbb{R}^3$ and r is referred to the position. The action of the group element on the function f is defined by the right group action

$$E(\Omega, v)f(P, r) = f(P\Omega, r\Omega + v). \tag{2}$$

The Haar measure on $\mathcal{E}_3$ is the product of the appropriate Haar measures of the rotation and translation groups. On the direction sphere S_2 the invariant surface element is

$$\mathrm{d}^2S(p) = n^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{3}$$

where n is the index of refraction and θ and ϕ are spherical angles.

In [3] a general strategy for constructing the configurational spaces of Euclidean Lie optics was considered. Two models of these spaces have been analysed. The first model, with elementary objects corresponding to light rays, leads to 4π geometrical optics and the second one, called Helmholtz optics in which the elementary objects are not lines but *planes* representing *wavefronts*, leads to scalar wave optics based on the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) F(r) + k^2 F(r) = 0 \tag{4}$$

where k denotes the wavenumber.

In Helmholtz optics [3, 6] the symmetry group of plane wavefronts is a two-dimensional Euclidean group $\mathcal{E}_2$. In this case the group of motion $\mathcal{E}_3$ can be decomposed in the following manner:

$$E(\Omega(\psi, \theta, \phi), r) = E(\Omega(\psi, 0, 0), (t_1, t_2, 0))E(\Omega(0, \theta, \phi), (0, 0, u) \tag{5}$$

where

$$(r_1, r_2, r_3) = (t_1, t_2, 0)\Omega(0, \theta, \phi) + (0, 0, u).$$

This factorization ensures that the coordinates (ψ, t_1, t_2) are in the appropriate coset (denoted by $\mathcal{H}^{\mathrm{wf}}$) while the coordinates (θ, ϕ, u) in the space $\mathcal{W} = \mathcal{E}_2 \backslash \mathcal{E}_3$. Instead of the angles θ and ϕ one can write a 3-vector p of fixed length n with the direction determined by (θ, ϕ). This vector is interpreted here as the optical momentum vector. A point in $\mathcal{W}$ can be interpreted as a wavefront, i.e. a plane $\mathcal{H}^{\mathrm{wf}}$ in the Euclidean space.

The space of all directions p (the optical momentum space) plays a crucial role in the Helmholtz optics and can be represented by a two-dimensional sphere S_2 (called the Descartes sphere) of radius n. On the Descartes sphere one can construct a space $L^2(S_2)$ of square-integrable functions $\Phi(p)$ on the sphere of all directions. The solutions of the optical Helmholtz equation (4) can be uniquely determined by the functions $\Phi(p)$ in the form:

$$F(r) = \frac{k}{2\pi n} \int_{S_2} \mathrm{d}^2 S(p)\, \Phi(p) \exp(\mathrm{i}kpr/n). \tag{6}$$

$F(r)$ is a continuous linear superposition of plane waves over all directions. Here and in the Helmholtz equation (4) k denotes the wavenumber. As we shall see later, the wavenumber k determines the appropriate representations of the Euclidean group $\mathcal{E}_3$ for Helmholtz Lie optics. Physically, fixing k models the Helmholtz optics as a monochromatic wavefield; for a more detailed discussion see section 6.9 of [3].

It will also be interesting to see how the action of the Euclidean group on $F(r)$ induces its action in the momentum space consisting of the functions $\Phi(p)$. In order to see this, one can use the integral representation (6) of the function $F(r)$

$$E(\Omega, r)F(r) = F(r\,\Omega + v) = \frac{k}{2\pi n} \int_{S_2} \mathrm{d}^2 S(p)\, \tilde{\Phi}(p) \exp(\mathrm{i}kpr/n) \tag{7}$$

and obtain the required transformation of the function $\Phi(p)$

$$\tilde{\Phi}(p) = (E(\Omega, v)\Phi)(p) = \Phi(p\Omega) \exp(\mathrm{i}k(p\Omega)v/n). \tag{8}$$

We exploit this action in the momentum space in further considerations.

The oscillatory solutions (6) of the Helmholtz equation were used in [3,6] to construct a possible space of wavefunctions for the wavefront Helmholtz optics. Geometric optics has no time variable and neither does the Helmholtz equation. 'Time' can be modelled as a distance measured along an arbitrary optical axis chosen here as the z axis. It has been pointed out [3] that every function $F(r)$ in (6) can be rewritten as a two-component function which satisfies the Helmholtz equation transformed to a Schrödinger-type equation with time $= z$. This way, by putting the coordinate $r_z = 0$, the three-dimensional optics can be reduced to two dimensions (on the screen). Using this two-component decomposition of the functions $F(r)$ one can build the Hilbert space $\mathcal{H}_k$ of the oscillatory solutions of the Helmholtz equation. The inner product of these functions should be unchanged if we move or rotate the screen to any other plane. Such a Euclidean invariant inner product was found by Steinberg and Wolf [4]. This inner product is non-local and has the form

$$(U, V)_{\mathcal{H}_k} = \int_{\mathbb{R}^3} \mathrm{d}^2 q \int_{\mathbb{R}^3} \mathrm{d}^2 q'\, U(q)^\dagger H_K(|q - q'|) V(q') \tag{9}$$

where q and q' are coordinates on the 'screen' and

$$U(q) = \begin{pmatrix} u(q, z)\,|_{z=0} \\ \dfrac{\partial u(q, z)}{\partial z}\,|_{z=0} \end{pmatrix}$$

$$H_K(|q - q'|) = \frac{1}{4\pi} \begin{pmatrix} k^2 \dfrac{j_1(k|q - q'|)}{k|q - q'|} & 0 \\ 0 & j_0(k|q - q'|) \end{pmatrix}.$$

This Hilbert space of the Helmholtz equation solutions $\mathcal{H}_k$ is unitarily equivalent to the Hilbert space $L^2(S_2)$ of functions in the optical momentum representation. In this paper we give an alternative construction for the Helmholtz Lie optics space, denoted by $\mathcal{K}$, which is unitarily equivalent to $L^2(S_2)$ and $\mathcal{H}_k$ spaces and consists of classes of scalar functions on three-dimensional space $\mathbb{R}^3$ instead of two-component functions on $\mathbb{R}^3$ as in $\mathcal{H}_k$.

3. AGCM

The goal of this section is to adapt the AGCM described in [5, 7] to the case of Euclidean Lie optics because in this special case, one needs to use the right action of the group of motion G instead of the left one. In this approach the main role is played by the $L^1(G)$ algebra with involution where G is the locally compact group of motions. In Helmholtz Lie optics it is the Euclidean group $G = \mathcal{E}_3$. The algebra $L^1(G)$ is the Banach algebra of complex functions on the group G with involution $\sharp$ given by the relation

$$u^{\sharp}(g) = u^*(g^{-1}) \tag{10}$$

where $*$ is the usual complex conjugation and the multiplication law is established by means of the right convolution in the following form

$$(u \star v)(g) = \int_G \mathrm{d}g'\, u(g')v(gg'^{-1}) \tag{11}$$

where $\mathrm{d}g$ denotes the Haar measure on G. Because the Euclidean group is unimodular the right and standard left convolution denoted in [5, 7] by $\circ$ are simply related to each other, namely

$$u \star v = v \circ u. \tag{12}$$

The second most important object in the AGCM approach is the functional (metastate) on the algebra $L^1(G)$ which describes the moving physical system. In our special case of Helmholtz (wavefront) Lie optics the wave packets describing wavefronts are moved by the Euclidean group. The most general metastate on $L^1(G)$ can be written in the form

$$\langle \rho; u \rangle = \int_G \mathrm{d}g\, u(g)\langle \rho; g \rangle \tag{13}$$

where $\langle \rho; g \rangle$ denotes a complex function on G which fulfils the following three conditions:

(i) $\langle \rho; e \rangle = 1$, where e denotes the unit element in the group G;

(ii) $\langle \rho; g^{-1} \rangle = \langle \rho; g \rangle^*$;

(iii) for any finite sequence $\alpha_1, \ldots, \alpha_n$ of complex numbers and any arbitrary sequence $g_1, g_2, \ldots, g_n$ of points on the group manifold, $n = 2, 3, \ldots$, the following relation is fulfilled

$$\sum_{i,j} \alpha_i^* \alpha_j \langle \rho; g_i^{-1} g_j \rangle \geqslant 0$$

(iv) the function defined by $\int_G \mathrm{d}g \langle \rho; g^{-1}g \rangle u(g)$ belongs to the algebra $L^1(G)$;

(v) the function $\langle \rho; g \rangle$ belongs to the space of essentially bounded functions usually denoted by $L^\infty(G)$.

We call the function $\langle\rho; g\rangle$ the metastate kernel (MK). Using the invariance of the Haar measure one can see, by direct calculations, that for an arbitrary function $\langle\rho; g\rangle \in L^1(G)$ condition (iv) is always fulfilled. Please note that we denote the functional (metastate) by the same symbol $\langle\rho; \rangle$ as the complex function $\langle\rho; g\rangle$ (MK) because it is easy to distinguish between them by specifying their arguments.

In the AGCM approach the space of quantum states is constructed by means of the well known GNS (Gelfand–Neumark–Segal) procedure in the algebraical approach to quantum mechanics and field theory [8, 9]. The functional (10) allows us to introduce a scalar product in the obtained linear space

$$(u|v) \equiv \langle\rho; u^\sharp \star v\rangle. \tag{14}$$

It is worthwhile to note that this scalar product is non-local with an integral kernel determined by the MK $\langle\rho; g\rangle$. The linear space obtained by the GNS procedure endowed with scalar product (14) can be turned into the Hilbert space $\mathcal{K}$ after the standard completion. As a result one obtains the Hilbert space of states represented by classes of the algebra elements indistinguishable from the group of motion for a fixed metastate. More precisely, the elements u and v belong to the same class when their difference has the following property:

$$\langle\rho; (u-v)^\sharp \star (u-v)\rangle = 0. \tag{15}$$

To find all these zero elements (15) is straightforwardly related to the zero eigenvalues problem of the overlap operator in the GCM approach [10]. Following this idea we introduce an analogue of the overlap operator (we will also call it the overlap operator) as follows

$$(\mathcal{N}^{\mathrm{R}}u)(g) = \int_G \mathrm{d}g' \langle\rho; g'g^{-1}\rangle u(g'). \tag{16}$$

One can prove that the element u of the algebra $L^1(G)$ is a zero element if and only if

$$\mathcal{N}^{\mathrm{R}}u = 0. \tag{17}$$

In [5, 7] the unitary equivalence between the standard GCM approach and the AGCM method has been shown for the standard form (left one) of the convolution operation, provided that the complex MK $\langle\rho; g\rangle$ is expressed in the form of a diagonal matrix element of a unitary representation $T(g)$ of the group G in a Hilbert space. For the right convolution (11) this correspondence is no longer unitary but anti-unitary. This statement can be directly proved in exactly the same manner as that used for the more standard case of left convolution [5, 7]. In a rough approximation, by choosing

$$\langle\rho; g\rangle = \langle\sim|T(g)|\sim\rangle \tag{18}$$

for every function $u \in L^1(G)$ one can find the corresponding wavepacket in the space of vectors $|\sim\rangle$:

$$u \to \int_G \mathrm{d}g\, u^*(g) T(g)^\dagger |\sim\rangle. \tag{19}$$

This way by using the AGCM one can construct the space of 'wave' states. In the next section we consider the details of this construction.

4. The optical state space for Helmholtz optics

As a first step we specify an elementary packet of wavefronts, a function $\Phi_0(p)$, as objects moved by the Euclidean group of motion. For this purpose, following the general ideas of the AGCM, we define the MK in the form of a scalar product determined in the optical momentum space $L^2(S_2)$ [3], namely

$$\langle \Phi_0; E(\Omega, v)\rangle_{S_2} = \frac{1}{n^2}(\Phi_0|E(\Omega, v)\Phi_0)_{S_2} = \frac{1}{n^2}\int_{S_2} \mathrm{d}^2S(p)\,\Phi_0(p)^* E(\Omega, v)\Phi_0(p) \tag{20}$$

where for the function Φ_0 we have chosen the simplest spherical harmonics $Y_{00}(p) = 1/\sqrt{4\pi}$. This choice of MK leads to a space describing the motion of the wavepackets in the form (19) with $|\sim\rangle = \Phi_0(p)$. A little algebra gives the following MK

$$\langle Y_{00}; E(\Omega, v)\rangle_{S_2} = \frac{1}{n^2}\int_{S_2} \mathrm{d}^2S(p)\, Y_{00}^*(p)Y_{00}(p\Omega)\exp(\mathrm{i}k(p\Omega)v/n). \tag{21}$$

The exponential function in equation (21) can be expanded in a series of spherical functions Y_{lm} and spherical Bessel functions j_l by the formula [11]:

$$\exp(\mathrm{i}r_1r_2) = 4\pi\sum_{l=0}^{\infty} \mathrm{i}^l j_l(|r_1||r_2|)\sum_{m=-l}^{l} Y_{lm}^*(\vartheta_1, \phi_1)Y_{lm}(\vartheta_2, \phi_2) \tag{22}$$

where (ϑ_k, ϕ_k), $k = 1, 2$, denotes the spherical angles of the vectors r_1 and r_2, respectively. Using the orthogonality of spherical harmonics, after straightforward integration we obtain a rather simple form for the MK, namely

$$\langle Y_{00}; E(\Omega, v)\rangle_{S_2} = j_0(k|v|). \tag{23}$$

Using this MK one can write the right overlap operator acting on a function $u \in L^1(G)$ in the form

$$(\mathcal{N}^{\mathrm{R}}u)(g') = \frac{1}{n^2}\int_{S_2} \mathrm{d}^2S(p)\int_{\mathcal{E}_3} \mathrm{d}g(E(g^{-1})\Phi_0(p))^*(E(g'^{-1})\Phi_0(p))u(g) \tag{24}$$

where we use the abbreviation $g = (\Omega, v)$. In our case of $\Phi_0(p) = Y_{00}(p)$ we get

$$(\mathcal{N}^{\mathrm{R}}u)(\Omega, v') = \int_{\mathcal{E}_3} \mathrm{d}^3v\,\mathrm{d}\Omega\, j_0(k|v - v'|)u(\Omega, v) \tag{25}$$

because

$$\int_{S_2} \mathrm{d}^2S(p)\exp(\mathrm{i}kp(v - v')/n) = 4\pi n^2 j_0(k|v - v'|). \tag{26}$$

The set $\mathcal{R}_{\Phi_0}$ of zero elements related to equation (15) (for more detailed consideration see [5, 7]), can now be found from equation (17). It is immediately seen that the set $\mathcal{R}_{\Phi_0}$ is a set of all functions $u(\Omega, v)$ which *do not contain the scalar part with respect to the $SO(3)$ group*. However, one can prove that *there are also functions which contain the scalar part with respect to the $SO(3)$ group and they belong to $\mathcal{R}_{\Phi_0}$*. It is seen from the relation between the metastate (13) with MK (23) and the scalar product in the optical momentum space $L^2(S_2)$

$$\langle Y_{00}; u^\sharp \star w\rangle_{S_2} = \frac{4\pi^3 n^2}{k^4}(\tilde{u}|\tilde{w})_{S_2} \tag{27}$$

where $\tilde{u}$ and $\tilde{w}$ denotes the usual Fourier transforms of the functions u and w with the following normalization

$$u(v) = \frac{k}{2\pi n}\int_{\mathbb{R}^3} \mathrm{d}^3 s\, \tilde{u}(s) \mathrm{e}^{\mathrm{i}ksv/n}. \tag{28}$$

This means that function u which is independent of Euler angles belongs to $\mathcal{R}_{\Phi_0}$ if and only if its Fourier transform $\tilde{u}$ given by (28), restricted to the Descartes sphere S_2, represents the zero vector in the optical momentum space $L^2(S_2)$.

To derive relation (27) it is sufficient to consider the functions u and w to be independent of Euler angles, i.e. one can start from the expression for the metastate (13) with MK (23). After some algebra it can be written as

$$\langle Y_{00}; u^\sharp \star w\rangle_{S_2} = \int_{\mathbb{R}^3} \mathrm{d}^3 r \int_{\mathbb{R}^3} \mathrm{d}^3 r'\, u^*(r) j_0(k|r - r'|) w(r'). \tag{29}$$

Now, making use of the integral form (26) of the spherical Bessel function j_0 for $v' = 0$, after changing the order of integration, one can easily obtain result (27).

In this way we have found the set $\mathcal{R}_{\Phi_0}$ of all functions u for which

$$\langle Y_{00}; u^\sharp \star u\rangle_{S_2} = 0. \tag{30}$$

The set $\mathcal{R}_{\Phi_0}$ is a left-ideal in the algebra $L^1(\mathcal{E}_3)$ and the space of states, following the GNS construction, is the completed quotient space

$$\mathcal{K} = L^1(\mathcal{E}_3)/\mathcal{R}_{\Phi_0} \tag{31}$$

with the scalar product

$$(\mathrm{cl}_{\mathcal{K}}(u)|\mathrm{cl}_{\mathcal{K}}(v))_{\mathcal{K}} \equiv \langle Y_{00}; u^\sharp \star v\rangle_{S_2} \tag{32}$$

where $\mathrm{cl}_{\mathcal{K}}(u)$ denotes the vector from the space $\mathcal{K}$ (i.e. the class of functions equivalent to the function u). Following the tradition of quantum mechanics of not using complicated notation, we usually write the representative u instead of the class $\mathrm{cl}_{\mathcal{K}}(u)$. Note that the scalar product (32) is non-local as in [4] and also Euclidean invariant.

From our considerations one can conclude that:

(i) every vector in the optical state space $\mathcal{K}$ is determined by its Fourier transform restricted to the Descartes sphere S_2;

(ii) this Fourier transform should belong to the optical momentum state space $\mathrm{L}^2(S_2)$; and

(iii) the correspondence between $\mathcal{K}$ and $\mathrm{L}^2(S_2)$ is unitary.

Now we may find the generators of the Eucildean group $\mathcal{E}_3$ on the state space $\mathcal{K}$ assuming the right action in the form

$$E(\Omega, v)\mathrm{cl}_{\mathcal{K}}(u) = \mathrm{cl}_{\mathcal{K}}(E(\Omega, v)u) \tag{33}$$

where

$$(E(\Omega, v)u)(r) = u(r\Omega + v). \tag{34}$$

Standard calculation shows that, in our case, the generators can be written in the same form as the very familiar generators in $L^2(\mathbb{R}^3)$. The operators

$$\hat{T}_x = \partial/\partial x \qquad \hat{T}_y = \partial/\partial y \qquad \hat{T}_x = \partial/\partial x \tag{35a}$$

generate translations in the x, y and z directions and $\hat{R}_k$ generate rotations around the appropriate axis

$$\hat{R}_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \qquad \hat{R}_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \qquad \hat{R}_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{35b}$$

The three-dimensional Euclidean algebra has two quadratic invariants, $\hat{T}^2 = \hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{R}}$. In our case the second invariant is identically equal to zero on $\mathcal{K}$. The first one can be calculated directly. For this purpose one can compute

$$(\hat{T}^2 + k^2)\mathrm{cl}_{\mathcal{K}}(u) = \mathrm{cl}_{\mathcal{K}}\left(\left(\frac{k}{2\pi n}\int_{\mathbb{R}^3} \mathrm{d}^3 s \left\{-\frac{k^2}{n^2}s^2 + k^2\right\} \tilde{u}(s)\mathrm{e}^{\mathrm{i}ksr/n}\right)\right) = \mathrm{cl}_{\mathcal{K}}(0) \tag{36}$$

because for $|s| = n$

$$\left\{-\frac{k^2}{n^2}s^2 + k^2\right\} \tilde{u}(s) = 0.$$

This interesting result implies that all vectors in $\mathcal{K}$ are eigenvectors of the first invariant

$$\hat{T}^2\mathrm{cl}_{\mathcal{K}}(u) = -k^2\mathrm{cl}_{\mathcal{K}}(u) \tag{37}$$

and they belong to the irreducible representation of the Euclidean group $\mathcal{E}_3$ determined by the wavenumber k. Because $\hat{T}^2$ is represented by the usual Laplace operator equation (37) represents Helmholtz equation (4). This means that every vector in $\mathcal{K}$ is a solution of the Helmholtz equation (4). This property can be expressed as follows. Every complex function on $\mathbb{R}^3$ which belongs to absolutely integrable functions space $L^1(\mathbb{R}^3)$ considered as an element of $\mathcal{K}$ is a solution of Helmholtz equation (4).

The space $\mathcal{K}$ can also be regarded as a space of states for wave Helmholtz optics. In [3] a wavization procedure has been considered by representing the generators of $\mathcal{E}_3$ in $\mathcal{H}_k$. However, within this procedure the multiplicative position operators $\hat{q}$ were not well defined. It seems that in our construction with $\mathcal{E}_3$ generators (35a) with the standard form, the position operators and the operators (35a) would fulfil the standard Heisenberg–Weyl commutation relations. In reality this is not the case. Operators like $\hat{q}$ are also not well defined in our space because their action in $\mathcal{K}$ depends on the choice of representatives from the classes(=vectors) $\mathrm{cl}_{\mathcal{K}}(u)$.

To find a condition which must be fulfilled by an operator A to be well posed in $\mathcal{K}$ one needs to consider its action on null functions, i.e. functions belonging to $\mathcal{R}_{\Phi_0}$. The action of A should be closed in the set $\mathcal{R}_{\Phi_0}$. One can express the null functions in the form

$$w(v) = \frac{k}{2\pi n}\int_{\mathbb{R}^3} \mathrm{d}^3 s(\tilde{w}(s) - \tilde{w}_{|S_2}(s))\mathrm{e}^{\mathrm{i}ksv/n} \tag{38}$$

where the subscript $|S_2$ denotes restriction to the Descartes sphere. The condition that the operator A should transform a null function into another one can thus be written as

$$\left[\int_{\mathbb{R}^3} \mathrm{d}^3 s' (\tilde{w}(s) - \tilde{w}_{|S_2}(s)) \left\{ \int_{\mathbb{R}^3} \mathrm{d}^3 v \mathrm{e}^{-\mathrm{i}ksv/n} A \mathrm{e}^{\mathrm{i}ks'v/n} \right\}\right]_{|S_2} = 0 \qquad (39)$$

Condition (39) is fulfilled by the generators (35), e.g. for the operators $\hat{T}_k$ from (35a) equation (39) is satisfied identically for every function $\tilde{w}$

$$[\tilde{w}(s) - \tilde{w}_{|S_2}(s)]_{|S_2} = 0.$$

Similar equations can be obtained for the rotation generators (35*b*). However, for standard position operators $\hat{q}_k$, with $k = x, y, z$, defined by $\hat{q}_k u(r) = r_k u(r)$, we get from (39) the following equation

$$\left[\frac{\partial}{\partial s_k}(\tilde{w}(s) - \tilde{w}_{|S_2}(s))\right]_{|S_2} = 0$$

which cannot be satisfied for all functions $\tilde{w}$.

The optical momenta operators which can be expressed by the generators (35*a*)

$$\hat{p}_k = \frac{-\mathrm{i}n}{k}\hat{T}_k \qquad (40)$$

are well defined and one can immediately see from equation (37) that they fulfil the required property for their ranges, i.e.

$$\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 = n^2. \qquad (41)$$

This implies that they have continuous and bounded spectra.

5. Summary

In this paper a new realization of the Helmholtz equation solutions space $\mathcal{K}$ has been considered. The space is a Hilbert space with a non-local and Euclidean invariant scalar product (32). It consists of classes of complex functions on the Euclidean $\mathcal{E}_3$ group manifold. The functions which differ only in the part which is non-scalar with respect to the $SO(3)$ group, belong to the same class—they represent the same vector in $\mathcal{K}$. However, among the $SO(3)$ scalar functions there are functions which belong to the same class if they have the same representations in the optical momentum space, i.e. every vector in $\mathcal{K}$ can be represented by an appropriate complex function on $\mathbb{R}^3$. All three spaces—$\mathcal{K}$, the optical momentum space $L^2(S_2)$ and the Helmholtz optics space $\mathcal{H}_k$—are unitarily equivalent but they offer different representations of the Helmholtz equation solutions space.

The space $\mathcal{K}$ regarded as a wavized Helmholtz optics space fulfils the conditions for bounded momenta; however, as in the case of previously obtained spaces [3], it does not admit the standard form of the position operator—maybe it does not exist at all. In general, searching for the appropriate observables in this model is still an unsolved problem which requires further investigation. The Euclidean group which is the group of motion for this model also provides a good and relatively simple structure allowing the investigation of some features which are helpful in the analysis of the Poincaré group as a group of motion. The case under consideration is also a non-trivial example of the application of the GNS construction.

By means of the AGCM formalism we also searched for the possibility of introducing vector as well as scalar Helmholtz optics but in vain. We obtained the empty space of states; see the appendix.

Appendix

An interesting question which naturally arises is to enquire about the possibility of constructing optical state space for vector instead of scalar Helmholtz Lie optics. To do this one will consider a function $\Phi_1(p)$ in the form of vector harmonics:

$$\Phi_1(p) \equiv Y^L_{JM}(\theta,\phi) = \sum_{m\sigma}(Jm1\sigma|JM)Y_{Jm}(\theta,\phi)\chi_\sigma \tag{A1}$$

where χ_σ denotes the spin matrix. The total angular momentum J consists of the orbital angular momentum L and spin equal to 1. Considering the Maxwell equations the transversity of electromagnetic waves has to be taken into account. This condition implies that $Y^L_{JM}(\theta,\phi)$ fulfils the requirement

$$rY^L_{JM}(\theta,\phi) = 0. \tag{A2}$$

The above equation imposes a certain condition on L, i.e. $L = J$. Following the idea of the AGCM method we define the MK in the form of a scalar product of vector spherical harmonics:

$$\langle Y^J_{JM}; E(\Omega,v)\rangle_{S_2} \equiv \frac{1}{n^2}\int_{S_2} \mathrm{d}^2S(p) Y^J_{JM}(p\Omega^{-1})^* Y^J_{JM}(p)\exp\left(\frac{ik}{n}pv\right). \tag{A3}$$

Using equations (A1) and (21) leads us to the result

$$\langle Y^J_{JM}; E(\Omega,v)\rangle_{S_2} = \sum_{m\sigma} |(Jm1\sigma|JM)|^2 \langle Y_{Jm}; E(\Omega,v)\rangle_{S_2}. \tag{A4}$$

Equation (A4) relates the new overlap operator $\mathcal{N}^{\mathrm{R}}_{Y^J_{JM}}$ determined by the MK (A3) to the overlap operator $\mathcal{N}^{\mathrm{R}}_{Y_{Jm}}$ given by the MK $\langle Y_{Jm}; E(\Omega,v)\rangle_{S_2}$. This relation can be symbolically written as

$$\mathcal{N}^{\mathrm{R}}_{Y^J_{JM}} = \sum_{m\sigma} |(Jm1\sigma|JM)|^2 \mathcal{N}^{\mathrm{R}}_{Y_{Jm}}. \tag{A5}$$

Using equation (25) and relation (A5) we get

$$(\mathcal{N}^{\mathrm{R}}u)(g') = \frac{1}{n^2}\sum_{m\sigma}|(Jm1\sigma|JM)|^2 \int_{S_2}\mathrm{d}^2S(p)\int_{\mathcal{E}_3}\mathrm{d}g[u(g)^*E(g^{-1})\Phi(p)]^*[E(g'^{-1})\Phi(p)]. \tag{A6}$$

After straightforward but tedious calculations one can show that the integral over S_2 is non-zero only if $J = 0$. This contradicts the triangle condition for the Clebsch–Gordan coefficients. Both contradictory conditions imply that the space of the vector Helmholtz Lie optics is trivial, i.e. it contains only the zero vector.

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